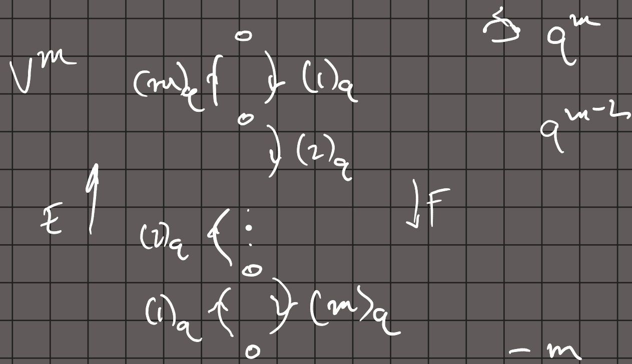
Math 261B Thurs. Nov. 19



$$(k)_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

$U_q(sl_2)$

$\Delta \longrightarrow V^\ell \otimes_\Delta V^m$ is a $U_q(sl_2)$

$V^1 \otimes V^1 \cong V^2 \oplus V^0$

<u>Theorem</u>: Over $A = \mathbb{Q}(q)$, every finite dimensional $\underset{U_q(sl_2)}{\wedge}$ module V which has <u>weight grading</u>: $V = \bigoplus V_k$ $\Leftarrow$ (i.e. V is an $\mathcal{O}_q(T)$ comodule)

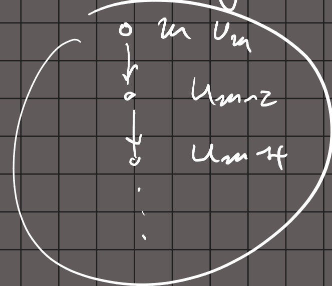
$K \mapsto q^k$ on V_k, is a $\bigoplus$ of various V^{m_i}, and V^m are irreducible.

Re: the proof: $\cdot$ V^m is irreducible

Suppose $0 \neq v \in V^m$ some $E^k v$ is a $\neq 0$ multiple of v_m

Now $F^j v_m$'s span V.

- Given V, let $v \neq 0$ be a weight vector of maximal weight, say m. Then E kills v.

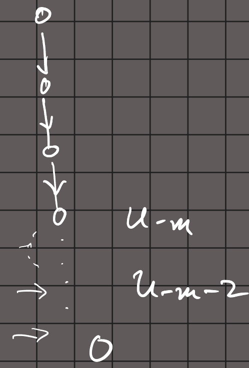


Let $u_{m-2k} = F^{(k)} v$ $\quad F^{(k)} = F^k / (k)_q!$

Can compute $E u_{m-2k}$ using

$$[E, F] = \text{"}(H)_q\text{"} = \frac{K - K^{-1}}{q - q^{-1}}$$

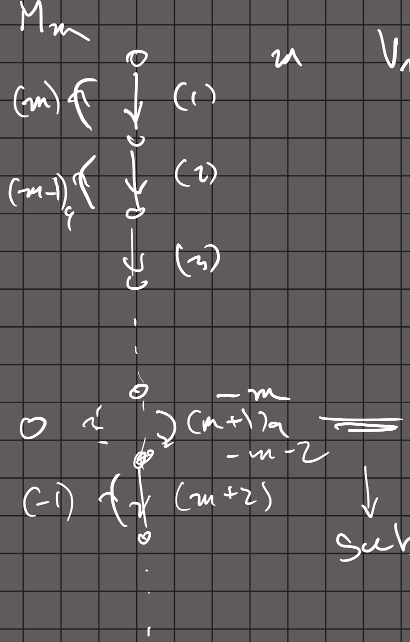
It's the same computation as last time.



$$E u_{m-2k} = (m-k+1)_q \, u_{m-2k-2}$$

$u_{m-2k-2} \neq 0 \Rightarrow u_{m-2k} \neq 0$ unless $k = m+1$

$\Rightarrow u_{-m-2} = 0$ $\quad u_{m-k} \neq 0$ for $k \leq m$ (V finite dimensional)



submodule

$V_m = M_m /$ unique max'l proper submodule

$\Rightarrow$ In any V we can find a V^m for some m

$\Rightarrow$ V^m's are all the irr reps.

$\Rightarrow$ V has a composition series with subquotients $\cong V^{m_i}$

$$V_{(0)} = 0 \subset V_{(1)} \subset V_{(2)} \subset \cdots \subset V_{(k)} = V$$

s.t. $V_{(i+1)}/V_{(i)} \cong V^{m_i}$

$\Rightarrow$ V has same weight space dimensions as same $\bigoplus V^{m_i}$

$$\chi_V(t) = \sum_k \dim(V_k)\, t^k \qquad \chi_{V^m} = t^m + t^{m-2} + \cdots + t^{-m}$$

$$= (m+1)_t$$

- Why is V a $\bigoplus$?

Classical casimir element

$$EF + FE + \frac{H(H-1)}{2} \in Z(\mathcal{U}(sl_2))$$

??

Lemme Quantum Casimir $\Theta = EF + FE + (2)_q \dfrac{K+K^{-1}}{(q-q^{-1})^2}$ is in the center of $\mathcal{U}_q(sl_2)$.

$(m)_q \; (\;) \; (1)_q$

– Check Θ commutes with E, F, K.

K is easy: $KEK^{-1} = q^2 E \quad KFK^{-1} q^{-2} E$

$KEFK^{-1} = q^0 EF = EF$

E $\quad [E, EF+FE] = E[E,F] + [E,F]E$

$$= E\frac{K-K^{-1}}{q-q^{-1}} + \frac{K-K^{-1}}{q-q^{-1}} E$$

$$= E\,\frac{(1+q^2)K - (1+q^{-2})K^{-1}}{q-q^{-1}}$$

$$\frac{(2)_q}{(q-q^{-1})^2}[E, K+K^{-1}] = \frac{(2)_q}{(q-q^{-1})^2} E\left((1-q^2)K + (1-q^{-2})K^{-1}\right)$$

$$E \cdot \frac{(2)_q}{q-q^{-1}}\left(-qK + q^{-1}K^{-1}\right)$$

$EK - KE$

$= (1-q^2) EK$

$$= E\left(\frac{-(1+q^2)K + (1+q^{-2})K^{-1}}{q-q^{-1}}\right)$$

Θ commutes with F by symmetry.

Θ acts on V^m as $\quad (m)_q + \dfrac{(2)_q\,(q^m+q^{-m})}{q-q^{-1}} = 2\,\dfrac{q^{m+1}+q^{-m-1}}{(q-q^{-1})^2}$

distinct for different m.

Split V into its Θ generalized eigenspaces:

$\Rightarrow$ Reduce to case V has only V^m (for one m) in its composition series, i.e. $\chi_V = d\cdot\chi_{V^m}$.

$\dim V_m = d$: pick basis $u^{(1)}, \dots, u^{(d)}$

Then $F^{(k)}u^{(i)}$ $\quad i=1,\dots,d \quad k=0,\dots,m$ are a basis of V, splitting it as $(V^m)^{\oplus d}$

We <u>assumed</u> the weight space decomposition — V is an $\mathcal{O}_q(T)$ comodule

Classical : $\mathcal{U}(\mathfrak{g})$

$\circlearrowleft$

V

$\mathfrak{g}$ semi-simple

$\Downarrow$

V has weight spaces

not true if G is reductive but not semisimple

$\mathfrak{g} = \mathrm{Lie}(G)$

e.g. $G = T$.

$V^m \otimes V^n$ $\quad m > n$

$V^1 \otimes V^1$

$$\chi_{V^m \otimes V^n} = \chi_{V^m} \cdot \chi_{V^n}$$

$$= \frac{q^{m+1} - q^{-m-1}}{q-q^{-1}} \cdot \frac{q^{n+1} - q^{-n-1}}{q-q^{-1}}$$

$$= \frac{q^{m+n+2} - q^{n-m} - q^{m-n} + q^{-m-n-2}}{(q-q^{-1})^2}$$

$$\frac{q^{m+n+2} - q^{m-n}}{q-q^{-1}} = q^{m+n+1} + q^{m+n-1} + \dots q^{m-n+1}$$

$Kv = q^k v$ $\qquad Kw = q^l w$

$\circlearrowleft$ $\qquad \circlearrowleft$

V $\qquad W$

in $V \otimes W$ $\qquad v \otimes w$

$$K \cdot (v \otimes w) = K \otimes K \, (v \otimes w)$$

$$= Kv \otimes Kw$$

$$= q^{k+l} \, v \otimes w$$

$$= \frac{q^{m+n+1} - q^{-(m+n+1)}}{q - q^{-1}} + \cdots + \frac{q^{m-n+1} - q^{-(m-n+1)}}{q-q^{-1}}$$

$$= \chi_{V^{m+n}} + \chi_{V^{m+n-2}} + \cdots + \chi_{V^{m-n}}$$

$$V^m \otimes V^n \cong V^{m+n} \oplus V^{m+n-2} \oplus V^{m+n-4} \oplus \cdots \oplus V^{m-n}$$

$$V^1 \otimes V^1 = V^2 \oplus V^0$$

This implies the subspace of $\mathcal{U}_q(\mathfrak{sl}_2)^*$ spanned by the matrix entries of all the V^m is a subalgebra $\mathcal{O}_q(SL_2)$

$A = \mathbb{Q}(q)$

$\mathcal{U}(\mathfrak{sl}_2)^*$

$$V^1 \otimes V^n = V^{n+1} \oplus V^{n-1}$$

$\Rightarrow$ Matrix entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of V^1 generate $\mathcal{O}_q(SL_2)$.

non-commutative

(Compare $\mathcal{O}(SL_2) = K[a,b,c,d]/(\det - 1)$)

Questions: What are the relations? What's the coproduct?

(Classically: have SL_2 $\mathcal{O}(SL_2)$ consists of functions on SL_2,

V^1 u $\qquad g \in SL_2$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

v

$$g u = (u\ v)\begin{pmatrix} a \\ c \end{pmatrix}$$

$$g v = (u\ v)\begin{pmatrix} b \\ d \end{pmatrix}$$

$$(u\ v) \overset{g}{\mapsto} (u\ v)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$g \in \mathcal{O}(SL_2)^*$

$\underbrace{}_{\text{comodule for } \mathcal{O}(SL_2)}$

$$u \mapsto u \otimes a + v \otimes c$$

$$v \mapsto u \otimes b + v \otimes d$$

$$(u\ v) \mapsto (u\ v) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

evaluate on

$x \in \mathcal{O}(SL_2)^*$

$x_1, x_2 \in U_q(sl_2)$ $\qquad x \in SL_2$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x_1) \qquad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

or $x \in \mathcal{U}(sl_2)$ ...

matrix of $x_1 x_2$ is $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$\uparrow$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_1 x_2) = \begin{pmatrix} \Delta a & \Delta b \\ \Delta c & \Delta d \end{pmatrix} (x_1 \otimes x_2)$$

$$\Delta a = a \otimes a + b \otimes c$$

$$\Delta b = a \otimes b + b \otimes d$$

$$\vdots$$

$n \times n$ matrix entries a_{ij} $\rightarrow$ $\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj}$

given by universal formulas from matrix multiplication.